Recognition of Some Symmetric Groups by the Set of the Order of Their Elements

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Abstract

For a finite group G, let $\pi_e(G)$ be the set of order of elements in G and denote S_n the symmetric group on n letters. We will show that if $\pi_e(G) = \pi_e(H)$, where H is S_p or S_{p+1} and p is a prime with $50 < p < 100$, then $G \cong H$.

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1 Introduction

For a finite group G, we denote by $\pi_e(G)$ the set of order of elements of G. If for a given finite group H, $\pi_e(G) = \pi_e(H)$ implies that $G \cong H$, then we say H is a *characterizable* group. There are scattered results in the literature showing that certain groups are characterizable. It is known that the symmetric groups S_n, $n = 2$, 3, 4, 5, 6, 8 are not characterizable, but the symmetric groups S_n, $n = 7$, 9, 11, 12, 13, 14 are characterizable (see [1], [3], [6] and [8]). In [4], the authors showed that the symmetric groups S_p and S_{p+1}, where $p \in \{23, 29, 31, 41, 43, 47\}$, are characterizable. In connection with these results, the following has been conjectured in [4]:

Conjecture *For all primes $p \geq 7$, the symmetric groups S_p and S_{p+1} (with the exception of S_8) are characterizable.*

In this paper we consider some symmetric groups and prove that they are characterizable. More precisely, the symmetric groups S_p and S_{p+1}, where $50 < p < 100$ is a prime, will be added to the list of groups with the above property.

Main Theorem *The symmetric groups S_p and S_{p+1}, where p is a prime and $50 < p < 100$, are characterizable.*

Throughout this paper, all groups are assumed to be finite and all simple groups are non-abelian. We denote by $\pi(G)$ the set of all prime divisors of the order of G. We use A_m and S_m to refer to the alternating and symmetric group of degree m,

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respectively. Further notations are standard and can be found, for example, in [2]. We will freely use the classification of all finite simple groups.

2 Preliminary Lemmas

The prime graph $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$ and two vertices $p,\ q \in \pi(G)$ are adjacent if and only if G contains an element of order pq. Denote by $t(G)$ the number of connected components of $\Gamma(G)$, and by $\pi_i = \pi_i(G)$; $i = 1,\ 2,\ \ldots,\ t(G)$; the connected components of $\Gamma(G)$. If $|G|$ is even, we will always assume that 2 is a vertex of π_1. We will use the following unpublished result of K. W. Gruenberg and O. H. Kegel (see [9, Theorem A]).

Lemma 2.1 *If G is a group whose prime graph has more than one component, then G has one of the following structures:*
(a) *Frobenius or* 2*-Frobenius,*

(b) *simple,*

(c) *an extension of a π_1-group by a simple group,*

(d) *simple by π_1,*

(e) *π_1 by simple by π_1.*

Note that in the above lemma, the group G is called 2-*Frobenius* if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ of G such that H is the Frobenius kernel of K and K/H is the Frobenius kernel of G/H.

Remark The Lemma 2.1 implies that, G has a normal series $1 \trianglelefteq N \triangleleft G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, G_1/N is a simple group, and G/G_1 is a solvable π_1-group.

Lemma 2.2 *Let G be a non-solvable group and $15 \in \pi_e(G)$. Then G is neither Frobenius nor* 2*-Frobenius.*

Proof. Since G is non-solvable, G cannot be 2-Frobenius. If G is a Frobenius group with kernel K and complement H, then H is non-solvable. Now by the structure theorem of non-solvable Frobenius complements (see [7, Theorem 18.6]), we know that H has a normal subgroup H_0 of index ≤ 2 such that $H_0 \cong SL(2,5) \times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \{2,\ 3,\ 5\} = \emptyset$. Since $\pi_e\big(SL(2,5)\big) = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 10\}$ and $15 \in \pi_e(G)$, we have $15 \in \pi_e(H) \setminus \pi_e(H_0)$. Therefore, there exists $1 \neq x \in H \setminus H_0$ such that $o(x) = 15$. Since $|H : H_0| = 2$, it follows that $x^2 \in H_0$ which implies that $o(x^2) = 15 \in \pi_e(H_0)$. This is a contradiction and the proof is complete. $\square$

We will utilize the following result concerning the set of order of elements of the symmetric groups (see [10]).

Lemma 2.3 *The group S_n (resp. A_n) has an element of order $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$, where $p_1,\ p_2, \ldots,\ p_s$ are distinct primes and $\alpha_1,\ \alpha_2, \ldots,\ \alpha_s$ are natural numbers, if and only if $p_1^{\alpha_1} + p_2^{\alpha_2} + \ldots + p_s^{\alpha_s} \leq n$ (resp. $p_1^{\alpha_1} + p_2^{\alpha_2} + \ldots + p_s^{\alpha_s} \leq n$ for m odd, and $p_1^{\alpha_1} + p_2^{\alpha_2} + \ldots + p_s^{\alpha_s} \leq n-2$ for m even).*

The following result of G. Higman will be used (see [5]):

Lemma 2.4 *Let G be a solvable group all of whose elements are of prime power order. Then $|\pi(G)| \leq 2$.*

We can now prove the following lemma which will be useful.

Lemma 2.5 *Let G be a group and $\pi_e(G) = \pi_e(S_p)$, where p is a prime number such that $p > 50$. Then G is non-solvable. Moreover, G is neither Frobenius nor 2-Frobenius.*

Proof. We know that $|S_p| = p! = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where p_i is the i-th prime, $p_k = p$ and for all i such that $p_i > (p-1)/2$, $\alpha_i = 1$. If G were solvable, there would exist more than 6 primes between $(p-1)/2$ and p (see [1]). So we can choose a $\{p_k,\ p_{k-1},\ p_{k-2}\}$-Hall subgroup H of G. Now by Lemma 2.3, S_p does not contain any element of order $p_k p_{k-1}$, $p_k p_{k-2}$ and $p_{k-1} p_{k-2}$. Thus all elements of H will be of prime power order. This contradicts Lemma 2.4 and so G must be non-solvable. The rest follows from Lemma 2.2. $\square$

Now the following result is in order.

Lemma 2.6 *Let G be a simple group. If $\{p\} \subset \pi(G) \subseteq \pi(p!)$, where p is a prime such that $50 < p < 100$, then G is isomorphic to one of the following groups:*

(*a*) *if* $p = 53$: $A_n,\ n = 53,\ 54,\ 55,\ 56,\ 57,\ 58;\ A_1(23^2),\ B_2(23),\ {}^2A_3(23);\ A_1(53)$,

(*b*) *if* $p = 59$: $A_n,\ n = 59,\ 60;\ A_1(59)$,

(*c*) *if* $p = 61$: $A_n,\ n = 61,\ 62,\ 63,\ 64,\ 65,\ 66;\ \ A_4(3^2),\ A_1(3^5),\ B_5(3),\ C_5(3);$ $A_3(11),\ A_1(11^2),\ A_2(11^2),\ B_2(11),\ B_3(11),\ C_3(11),\ D_4(11),$ ${}^2A_3(11);\ A_2(13),\ A_3(13);\ A_2(47),\ A_3(47);\ A_1(61)$,

(*d*) *if* $p = 67$: $A_n,\ n = 67,\ 68,\ 69,\ 70;\ Ly;\ A_2(29);\ A_2(37),\ A_1(37^3),\ G_2(37);$
$A_1(67),$

(*e*) *if* $p = 71$: $A_n,\ n = 71,\ 72;\ F_1;\ A_4(5),\ A_5(5);\ A_1(71),$

(*f*) *if* $p = 73$: $A_n,\ n = 73,\ 74,\ 75,\ 76,\ 77,\ 78;\ A_2(2^3),\ A_1(2^9),\ G_2(2^3),\ C_4(2^3),$
$E_6(2),\ B_3(2^2),\ C_3(2^3);\ A_1(3^6),\ A_5(3^2),\ B_6(3),\ B_2(3^3),\ C_6(3),$
$D_4(3^2),\ G_2(3^2),\ F_4(3),\ {}^2A_2(3^2),\ {}^2A_3(3^3),\ {}^2D_6(3),\ {}^2E_6(3);$
$A_1(73),\ A_1(73^2),\ B_2(73),$

(*g*) *if* $p = 79$: $A_n,\ n = 79,\ 80,\ 81,\ 82;\ A_2(23),\ A_3(23),\ A_2(23^2),\ A_1(23^3),$
$B_3(23),\ C_3(23),\ D_4(23),\ G_2(23);\ A_1(79),\ A_2(79),$

(*h*) *if* $p = 83$: $A_n,\ n = 83,\ 84,\ 85,\ 86,\ 87,\ 88;\ A_1(83),\ A_1(83^2),$

(*i*) *if* $p = 89$: $A_n,\ n = 89,\ 90,\ 91,\ 92,\ 93,\ 94,\ 95,\ 96;\ A_1(89),$

(*j*) *if* $p = 97$: $A_n,\ n = 97,\ 98,\ 99,\ 100;\ A_2(61);\ A_1(97).$

Proof. Let $p = 53$. Then $\{53\} \subset \pi(G) \subseteq \pi(53!)$. First, suppose G is an alternating group A_n. We know that $\frac{1}{2}(n!)$ is divisible by 53 if and only if $n \geq 53$. This implies that the only possibilities for n are 53, 54, 55, 56, 57 or 58. Next, suppose G is a sporadic simple group. The orders of sporadic simple groups are listed in [2]. From this, it is easy to see that G cannot be isomorphic to a sporadic simple group. Finally, we suppose that G is a group of Lie type over a finite field of order $q = r^s$ (r prime). Since r divides $|G|$, r must be equal to $2,\ 3,\ 5, \ldots,\ 51$ or 53. Now suppose $r = 2$. Since the order of 2 modulo 53 is 52, and $53 | 2^k - 1$ and $2^k - 1 \mid |G|$ it follows that k is a multiple of 52. From [2, Table 6], we see that no candidate for G exists in this case. Similarly for $r \in \pi(53!) \backslash \{23,\ 53\}$ we do not get a group. Suppose $r = 23$. Since the order of 23 modulo 53 is 4, from [2, Table 6], the only candidates from G are $A_1(23^2)$, $B_2(23)$ and ${}^2A_3(23)$. Finally we assume that $r = 53$. Similarly, in this case the only possible group is $A_1(53)$.

The rest of the proof is repetition of similar processes. □

We will need the following results (see [10]).

Lemma 2.7 *If S_m contains a Frobenius subgroup $N\langle\gamma\rangle$ with kernel N of order n and cyclic complement $\langle\gamma\rangle$ of order p^k, where p is a prime, then for every natural power $t = p^r$ of p there is a Frobenius subgroup in S_{mt} with kernel of order n^t and cyclic complement of order $tp^k = p^{k+r}$.*

Remark If p is odd prime, the Frobenius group stated in Lemma 2.7 lies in A_{mt}. This is also true for $p = 2$ provided that $r \geq 1$ and, the number m together with the permutation γ are both even or odd.

Lemma 2.8 *Let N be a normal elementary abelian p-subgroup of G, $K \cong G/N$ and let $G_1 = NK$ be the natural semidirect product. Then $\pi_e(G_1) \subseteq \pi_e(G)$.*

According to the discussion of A. V. Zavarnitsin and V. D. Mazurov concerning the action of an alternating and symmetric group on an elementary abelian p-group in [10, Propositions 2 and 3] we have the following result:

Lemma 2.9 *Let p be an odd prime and V be an elementary abelian p-group, on which the group A_m $(m \geq 3)$ acts. Then $\pi_e(VA_m) \nsubseteq \pi_e(S_m)$, where VA_m is the natural semidirect product.*

We will need the following.

Lemma 2.10 (see [6]) *Let G be a group, N a normal subgroup of G and G/N a Frobenius group with Frobenius kernel F and cyclic complement C. If $(|F|,\ |N|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime p of $|N|$.*

Lemma 2.11 (see [10]) *Let G be a semidirect product of an elementary abelian p-group V with A_m. Suppose that A_m contains Frobenius subgroups $F_1,\ F_2, \dots,\ F_t$ with cyclic complements $\langle d_i \rangle$ of orders $p_i^{s_i}$, where p_i's are pairwise distinct primes, such that the supports,* supp F_i, *are pairwise disjoint. Moreover, suppose that the order of the kernel of each F_i is prime p, and, all $a_i = |\text{supp } F_i|$ are distinct. Then $p\ o(d_1) \dots o(d_t) \in \pi_e(G)$.*

We now give the following.

Lemma 2.12 *Let G be a group and $\pi_e(G) \subseteq \pi_e(S_p)$ (resp. $\pi_e(S_{p+1})$) where p is a prime such that $50 < p < 100$. Moreover, assume that G possesses a normal elementary abelian r-subgroup N such that $G/N \cong A_p$ (resp. A_{p+1}). Then $N = 1$.*

Proof. Assume that $N \neq 1$. It is clear that from Lemma 2.8 that $\pi_e(NA_p) \subseteq \pi_e(G) \subseteq \pi_e(S_p)$. On the other hand, Lemma 2.9 forces $r = 2$. Now, we deal with each prime number separately.

(*i*) $p \in \{53,\ 59,\ 61\}$: Since S_9 contains a Frobenius subgroup of order 8×9 with cyclic complement of order 8, by Lemma 2.7, it follows that S_{36} has a Frobenius subgroup with cyclic complement of order 32. Clearly $S_{36} < A_p$. By Lemma 2.10, we conclude that $64 \in \pi_e(G)$, a contradiction.

(*ii*) $p \in \{71,\ 73,\ 79,\ 83,\ 89,\ 97\}$: Since S_{17} contains a Frobenius subgroup of order 16×17 with cyclic complement of order 16, by Lemma 2.7, S_{68} contains a Frobenius subgroup with cyclic complement of order 64. On the other hand, $S_{68} < A_p$ and so by Lemma 2.10 we obtain $128 \in \pi_e(G)$, again a contradiction.

(*iii*) p=67: As discussed before, S_{36} contains a Frobenius subgroup of order 32×9^4 with cyclic complement of order 32. Moreover, $S_{36} < A_p$. On the other hand, A_{11} contains a Frobenius subgroup of order 5×11, and $A_{11} < A_p$. Therefore, A_p has a pair of Frobenius subgroups of orders 32×9^4 and 5×11 satisfying the hypothesis of Lemma 2.11, and hence $5 \times 64 \in \pi_e(G)$. This contradiction completes the proof of the lemma. $\square$

3 Proof of the Main Theorem

Suppose G is a group such that $\pi_e(G) = \pi_e(S_p)$, where p is a prime with $50 < p < 100$. The components of the prime graph of G are

$$\pi_1(G) = \{2,\ 3,\ 5, \ldots,\ r_{p-1}\} \quad \text{and} \quad \pi_2(G) = \{p\},$$

where r_{p-1} denotes the largest prime not exceeding $p-1$. We must prove that G is isomorphic to S_p. By Lemma 2.5, G is neither Frobenius nor 2-Frobenius. Next, from Lemma 2.1, G has a normal series $1 \trianglelefteq N \triangleleft G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, $\overline{G}_1 := G_1/N$ is simple, and G/G_1 is a solvable π_1-group. Furthermore, we have

$$\frac{G}{N} \cong \frac{N_{G/N}(\overline{G}_1)}{C_{G/N}(\overline{G}_1)} \hookrightarrow \mathrm{Aut}(\overline{G}_1).$$

Thus we may assume that $G/N \leq \mathrm{Aut}(\overline{G}_1)$. Note that $\{2,\ p\} \subseteq \pi(\overline{G}_1) \subseteq \pi(G) = \pi(p!)$, $\pi_1(\overline{G}_1) \subseteq \pi_1(G) = \{2,\ 3, \ldots,\ r_{p-1}\}$ and $\pi_2(\overline{G}_1) = \pi_2(G) = \{p\}$. Hence $\overline{G}_1$ is a simple group such that $p \mid |\overline{G}_1|$. Now we claim that $\overline{G}_1 \cong A_p$.

According to the classification of finite simple groups, we know that the possibilities for $\overline{G}_1$ are the alternating groups A_n, with $n \geq 5$, sporadic simple groups, and finite groups of Lie type. We deal with the above cases separately.

First, suppose $\overline{G}_1$ is a simple group of Lie type or sporadic. By Lemma 2.6, $\overline{G}_1$ can only be isomorphic to one of the listed simple groups of Lie type or sporadic. For each prime p appearing in (a) – (j) of Lemma 2.6, there are two other primes $r_1,\ r_2 \in \pi(G) \backslash \pi\big(\mathrm{Aut}(\overline{G}_1)\big)$ with $r_1 + r_2 > p$, where $\overline{G}_1$ is any simple group on the appropriate list:

p	53	59	61	67	71	73	79	83	89	97
r_1	43	47	53	59	61	67	71	73	79	83
r_2	47	53	59	61	67	71	73	79	83	89

Moreover, from $G/N \leq \mathrm{Aut}(\overline{G}_1)$, it follows that $r_1,\ r_2 \in \pi(N)$, and the nilpotency of N forces $r_1r_2 \in \pi_e(N) \subset \pi_e(G) = \pi_e(S_p)$, which is a contradiction, because $r_1 + r_2 > p$.

Next, suppose $\overline{G}_1$ is a simple group of alternating type listed in Lemma 2.6. Then in all cases, there exist primes q_1 and q_2 such that $q_1 + q_2 = p + 1$:

p	53	59	61	67	71	73	79	83	89	97
q_1	7	7	3	7	5	3	7	5	7	19
q_2	47	53	59	61	67	71	73	79	83	79

But then $q_1q_2 \in \pi_e(A_n) \setminus \pi_e(G)$ for $n > p$, by Lemma 2.3. Therefore, $\overline{G}_1 \ncong A_n$ for $n > p$. Hence $\overline{G}_1 \cong A_p$ and the claim has been proved.

Finally, we show that $G \cong S_p$. By Lemma 2.12, $N = 1$. On the other hand, since $G_1 \leq G \leq \mathrm{Aut}(G_1)$ and $|\mathrm{Out}(G_1)| = 2$, it follows that $G \cong A_p$ or $G \cong S_p$. But since $\pi_e(A_p) \neq \pi_e(S_p)$, we have $G \cong S_p$.

Next, suppose G is a group such that $\pi_e(G) = \pi_e(S_{p+1})$, where p is a prime with $50 < p < 100$. By using similar arguments to those in the proof of the previous case, G has a normal series $1 \trianglelefteq N \triangleleft G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, $\overline{G}_1 := G_1/N$ is simple, and G/G_1 is a solvable π_1-group. We may assume that $G/N \leq \mathrm{Aut}(\overline{G}_1)$. Now we claim that $\overline{G}_1 \cong A_{p+1}$.

The proof of this claim is similar to those in the proof of the previous case. For example, we proceed for $p = 53$. Using similar arguments to those in the proof of the previous case, we see that $\overline{G}_1 \cong A_{53}$ or A_{54}. If $\overline{G}_1$ is isomorphic to A_{53}, then since $|\mathrm{Out}(A_{53})| = 2$ and $7 \times 47 \in \pi_e(G) \setminus \pi_e(A_{53})$, it follows that $N \neq 1$ and $7 \in \pi(N)$ or $47 \in \pi(N)$. Let N be a 7-group. Then $\overline{G}_1$ has a pair of Frobenius subgroups of orders 32×9^2 and 7×8 satisfying the hypothesis of Lemma 2.11, and hence $32 \times 49 \in \pi_e(G)$. This is a contradiction. Thus N is a 47-group. Yet, in this case, $\overline{G}_1$ contains a Frobenius subgroup of order 8×9 with cyclic complement of order 8, and by Lemma 2.10 we get $8 \times 47 \in \pi_e(G)$, which contradicts the assumption. Therefore, $\overline{G}_1 \cong A_{54}$.

The rest of the proof is similar to the proof in the previous case and we have $G \cong S_{p+1}$. $\square$

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